Boundary-type meshless solutions of potential problems: Comparison between singular and regular formulations in Hybrid BNM

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第20回境界要素法シンポジウム

Introduction

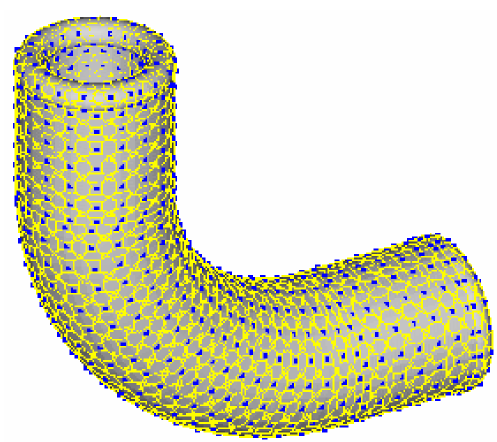
	Domain type	Boundary type
Pseudo meshless	Element free Galerkin method	Boundary node method
Truly meshless	Meshless Local Boundary Integral Equation	?

**In FEM and BEM, elements are used for two purposes: (i) Variables interpolation; (ii) numerical integration

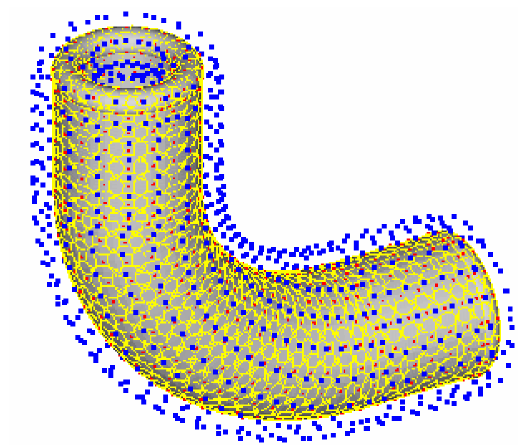
Introduction(2)

➢Singular Hybrid BNM & Regular Hybrid BNM

- Combines a modified functional with the *Moving Least Squares* (MLS) approximation
- Boundary-only meshless method

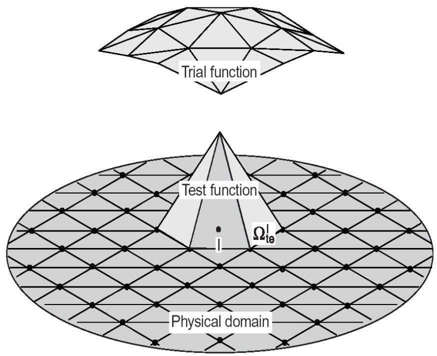


Example of SHBNM discretization

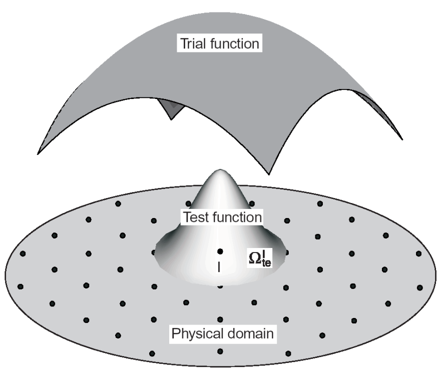


Example of RHBNM discretization

MLS Approximation



Element interpolation



MLS approximation

$$\boxed{\tilde{u}(\mathbf{s}) = \sum_{I=1}^{N} \Phi_I(\mathbf{s}) \hat{u}_I}$$

$$\boxed{\Phi_{I,k} = \sum_{j=1}^{m} [p_{j,k} (A^{-1}B)_{jI} + p_j (A^{-1}B_{,k} + A_{,k}^{-1}B)_{jI}]}$$

$$\boxed{A_{,k}^{-1} = -A^{-1} A_{,k} A^{-1}}$$

where

$$\Phi_I(\mathbf{s}) = \sum_{j=1}^{m} p_j(\mathbf{s}) \left[A^{-1}(\mathbf{s}) B(\mathbf{s}) \right]_{jI}$$

$$A(\mathbf{s}) = \sum_{I=1}^{N} w_I(\mathbf{s}) \mathbf{p}(\mathbf{s}^I) \mathbf{p}^{\mathbf{T}}(\mathbf{s}^I)$$

$$B(\mathbf{s}) = \left[w_1(\mathbf{s}) \mathbf{p}(\mathbf{s}^1), w_2(\mathbf{s}) \mathbf{p}(\mathbf{s}^2), \cdots, w_N(\mathbf{s}) \mathbf{p}(\mathbf{s}^N) \right]$$

SHBNM & RHBNM

- **Modified variational principle**
  - Modified functional

$$\Pi_{AB} = \int_\Omega \frac{1}{2} u,_i \, u,_i \, d\Omega - \int_\Gamma \tilde{q}(u - \tilde{u}) d\Gamma - \int_{\Gamma_q} \bar{q}\tilde{u} \, d\Gamma$$

  - Three independent variables

Boundary potential; Boundary normal flux; Potential in the domain.

- **Variables approximation**
  - Domain variables

$$u = \sum_{I=1}^{N} u_I^s x_I \qquad u_I^s = \frac{1}{\kappa} \frac{1}{4\pi r(Q, P_I)}$$

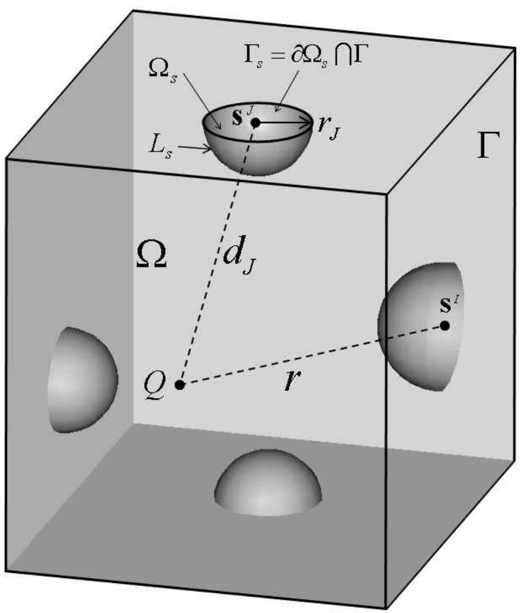
  - Boundary variables

$$\tilde{u}(\mathbf{s}) = \sum_{I=1}^{N} \Phi_I(\mathbf{s}) \hat{u}_I \qquad \tilde{q}(\mathbf{s}) = \sum_{I=1}^{N} \Phi_I(\mathbf{s}) \hat{q}_I$$

SHBNM & RHBNM(2)

- **Local weak form**

$$\int_{\Gamma}(q-\tilde{q})\delta u d\Gamma-\int_{\Omega}u_{,ii}\delta u d\Omega+\int_{\Gamma_q}(\tilde{q}-\bar{q})\delta\tilde{u}d\Gamma-\int_{\Gamma}(u-\tilde{u})\delta\tilde{q}d\Gamma=0$$

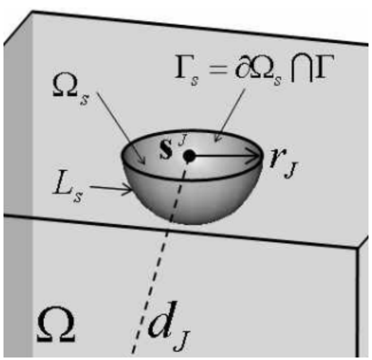


$$\int_{\Gamma_s+L_s}(q-\tilde{q})\delta u d\Gamma-\int_{\Omega_s}u_{,ii}\delta u d\Omega=0$$
$$\int_{\Gamma_s+L_s}(u-\tilde{u})\delta\tilde{q}d\Gamma=0$$

SHBNM & RHBNM(3)

$$\int_{\Gamma_s + L_s} (q - \tilde{q}) v \, d\Gamma - \int_{\Omega_s} u_{,ii} v \, d\Omega = 0$$
$$\int_{\Gamma_s + L_s} (u - \tilde{u}) v \, d\Gamma = 0$$

$$v_J(Q) = \begin{cases} \dfrac{\exp[-(d_J/c_J)^2] - \exp[-(r_J/c_J)^2]}{1 - \exp[-(r_J/c_J)^2]}, & r_J \geq d_j \geq 0 \\ 0, & d_J \geq r_J \end{cases}$$



$$\int_{\Gamma_s} (q - \tilde{q}) v \, d\Gamma - \int_{\Omega_s} u_{,ii} v \, d\Omega = 0$$
$$\int_{\Gamma_s} (u - \tilde{u}) v \, d\Gamma = 0$$

$$u = \sum_{I=1}^{N} u_I^s x_I$$
$$\tilde{u}(\mathbf{s}) = \sum_{I=1}^{N} \Phi_I(\mathbf{s}) \hat{u}_I$$
$$\tilde{q}(\mathbf{s}) = \sum_{I=1}^{N} \Phi_I(\mathbf{s}) \hat{q}_I$$

SHBNM & RHBNM(4)

➢Singular formulation

$$\frac{1}{2}x_J + \sum_{I=1}^{n}\int_{\Gamma_s}\frac{\partial u_I^s}{\partial n}v_J(Q)x_I d\Gamma = \sum_{I=1}^{n}\int_{\Gamma_s}\Phi_I(\mathbf{s})v_J(Q)\hat{q}_I d\Gamma$$

$$\sum_{I=1}^{n}\int_{\Gamma_s}u_I^s v_J(Q)x_I d\Gamma = \sum_{I=1}^{n}\int_{\Gamma_s}\Phi_I(\mathbf{s})v_J(Q)\hat{u}_I d\Gamma$$

$$\mathbf{Ux} = \mathbf{H\hat{q}} \qquad \mathbf{Vx} = \mathbf{H\hat{u}}$$

$$P_I = \mathbf{s}_I$$

$$\mathbf{UV}^{-1}\mathbf{H\hat{u}} - \mathbf{H\hat{q}} = 0$$

➢Regular formulation

$$\sum_{I=1}^{n}\int_{\Gamma_s}\frac{\partial u_I^s}{\partial n}v_J(Q)x_I d\Gamma = \sum_{I=1}^{n}\int_{\Gamma_s}\Phi_I(\mathbf{s})v_J(Q)\hat{q}_I d\Gamma$$

$$\sum_{I=1}^{n}\int_{\Gamma_s}u_I^s v_J(Q)x_I d\Gamma = \sum_{I=1}^{n}\int_{\Gamma_s}\Phi_I(\mathbf{s})v_J(Q)\hat{u}_I d\Gamma$$

$$\mathbf{Ux} = \mathbf{H\hat{q}} \qquad \mathbf{Vx} = \mathbf{H\hat{u}}$$

$$P_I = \mathbf{s}_I + \mathbf{n}(\mathbf{s}_I)\cdot h \cdot SF$$

$$\mathbf{Ax} = \mathbf{d}$$

Secondary results

➢Singular HBNM

- At boundary

$$\begin{Bmatrix} \tilde{q} \\ \partial\tilde{u}/\partial s_1 \\ \partial\tilde{u}/\partial s_2 \end{Bmatrix} = \begin{bmatrix} n_1 & n_2 & n_3 \\ \partial x_1/\partial s_1 & \partial x_2/\partial s_1 & \partial x_3/\partial s_1 \\ \partial x_1/\partial s_2 & \partial x_2/\partial s_2 & \partial x_3/\partial s_2 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}$$

$$\partial\tilde{u}/\partial s_k = \sum_{I=1}^{N} \Phi_{I,k}\hat{u}_I$$

- At internal points

$$\begin{aligned} u(P) &= \int_\Gamma U(Q,P)\tilde{q}(Q)d\Gamma - \int_\Gamma \frac{\partial U(Q,P)}{\partial n(Q)}\tilde{u}(Q)d\Gamma \\ &= \sum_{panels}\int_{\Gamma^p} U(Q,P)\tilde{q}(Q)d\Gamma - \sum_{panels}\int_{\Gamma^p} \frac{\partial U(Q,P)}{\partial n(Q)}\tilde{u}(Q)d\Gamma \\ q_i(P) &= \int_\Gamma \frac{\partial U(Q,P)}{\partial x_i(P)}\tilde{q}(Q)d\Gamma - \int_\Gamma \frac{\partial^2 U(Q,P)}{\partial x_i(P)\partial n(Q)}\tilde{u}(Q)d\Gamma \\ &= \sum_{panels}\int_{\Gamma^p} \frac{\partial U(Q,P)}{\partial x_i(P)}\tilde{q}(Q)d\Gamma - \sum_{panels}\int_{\Gamma^p} \frac{\partial^2 U(Q,P)}{\partial x_i(P)\partial n(Q)}\tilde{u}(Q)d\Gamma \end{aligned}$$

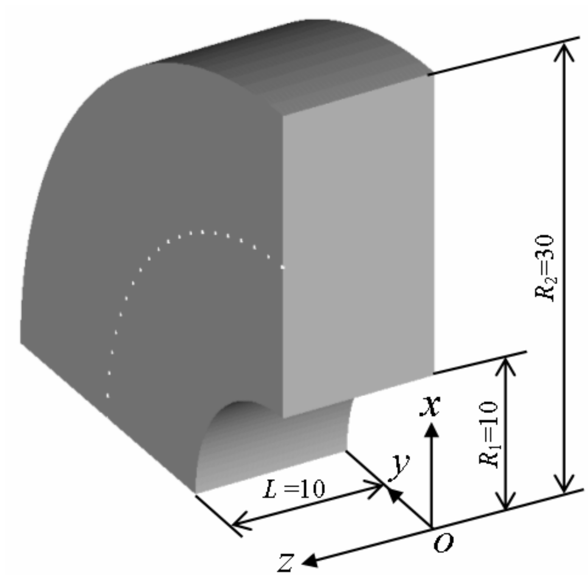
➢Regular HBNM

- At both boundary & internal points

$$u = \frac{1}{4\pi\kappa}\sum_{I=1}^{N}\frac{1}{r(Q,P_I)}x_I$$

$$q_i = \frac{1}{4\pi}\sum_{I=1}^{N}\frac{x_i(Q)-x_i(P_I)}{r^3(Q,P_I)}x_I$$

Numerical results



Geometry and dimensions

- Analytical solution

$$u = x^3 + y^3 + z^3 - 3yx^2 - 3xz^2 - 3zy^2$$

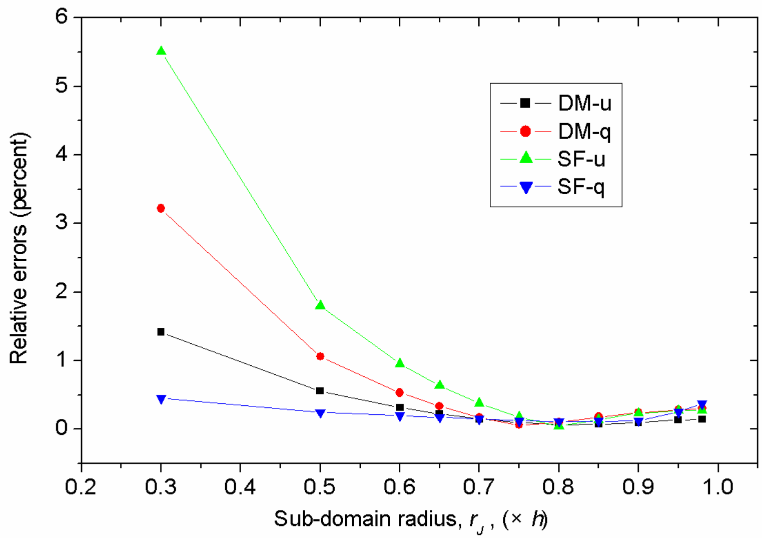
- Error estimation

$$e = \frac{1}{|u|_{\max}} \sqrt{\frac{1}{N} \sum_{i=1}^{N} \left(u_i^{(e)} - u_i^{(n)} \right)^2}$$

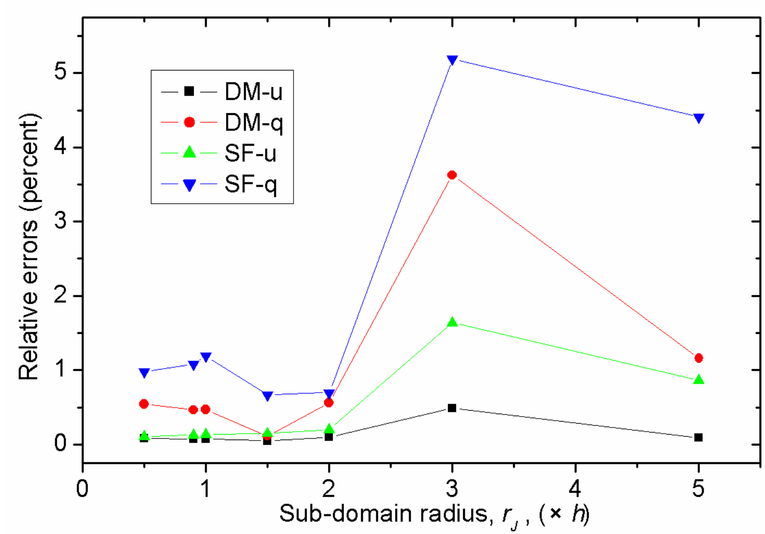
DM-u: Errors for potentials inside domain.
DM-q: Errors for fluxes inside domain.
SF-u: Errors for potentials at boundary.
SF-q: Errors for fluxes at boundary.

Numerical results (2)

- **Optimal value for sub-domain radius, r_J**



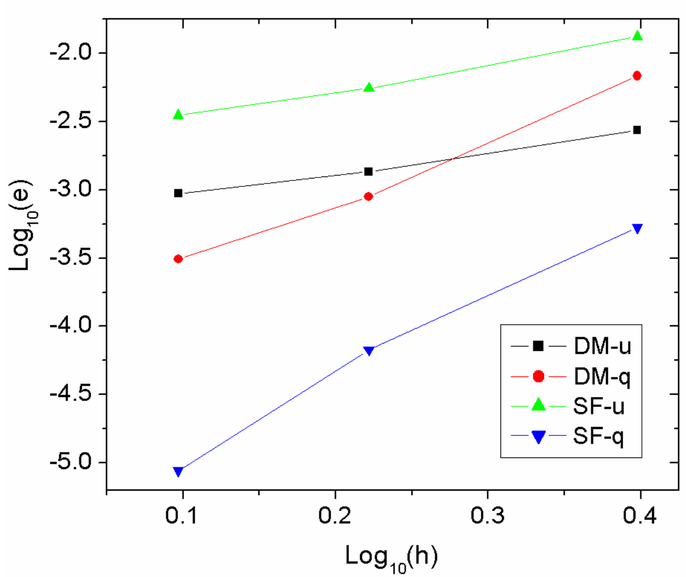
Singular HBNM, 1974 nodes used



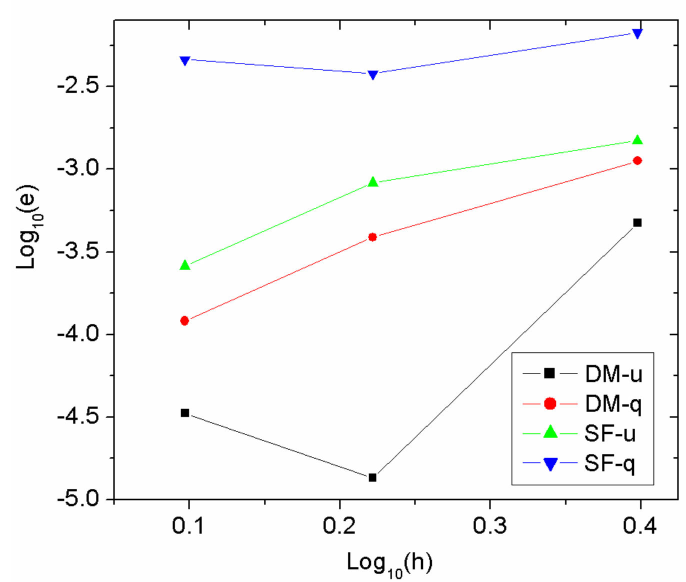
Regular HBNM, 504 nodes used

Numerical results (3)

- **Convergence rate**



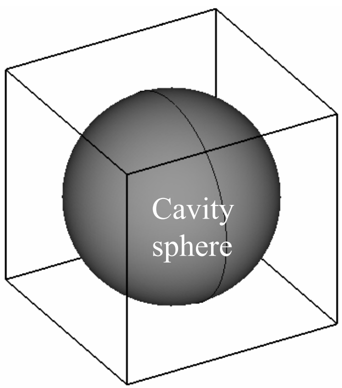
Singular HBNM

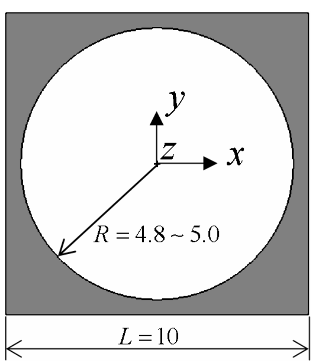


Regular HBNM

Numerical results (4)

- **Applicability to thin structures** (a cube containing a spherical cavity)





Singular Hybrid BNM

R	4.8	4.9	4.99	4.999	4.9999
Error-u (%)	0.0524	0.259	1.108	23.43	fail
Error-q (%)	0.714	1.185	7.346	141.8	fail

Regular Hybrid BNM

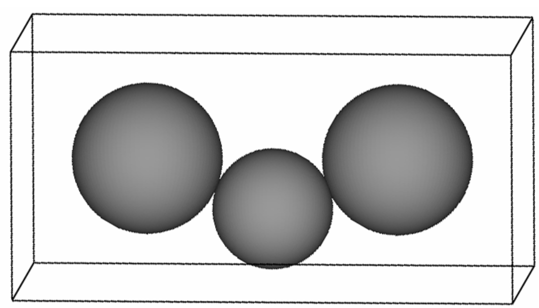
R	4.8	4.9	4.99	4.999	4.9999	5.0
Error-u (%)	0.01010	0.00885	0.00806	0.00800	0.00800	0.00800
Error-q (%)	0.04259	0.04262	0.04257	0.04251	0.04250	0.04250

Concluding remarks

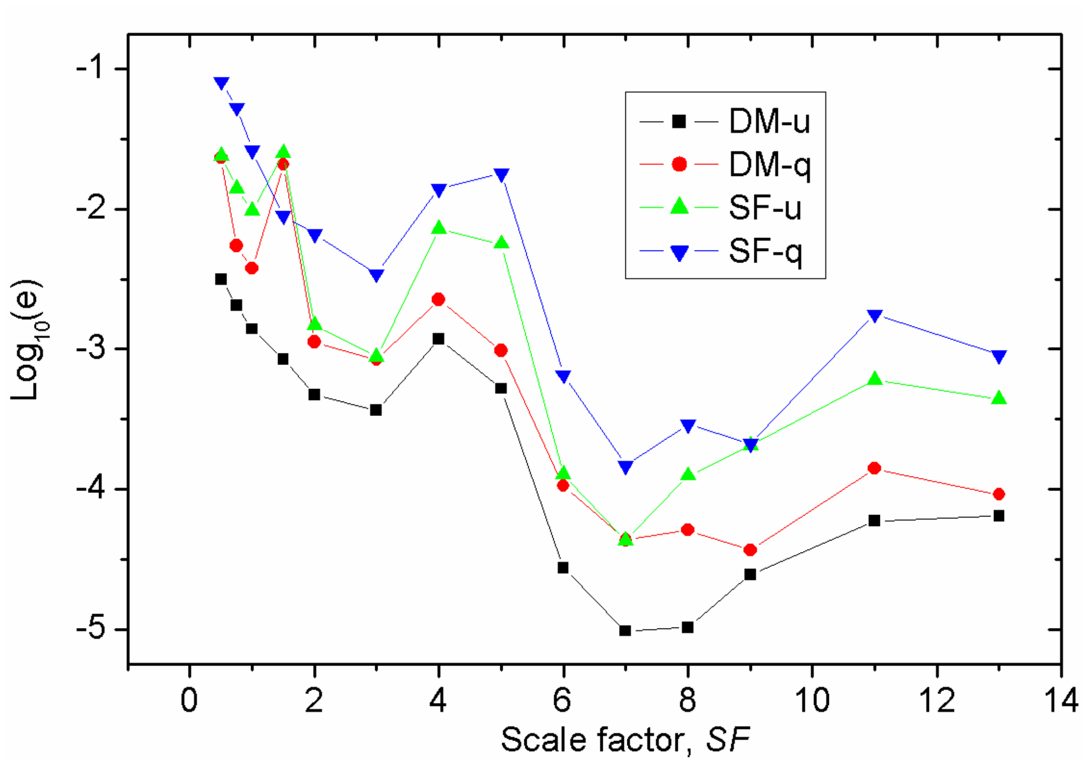
- SHBNM and RHBNM have been compared in several aspects

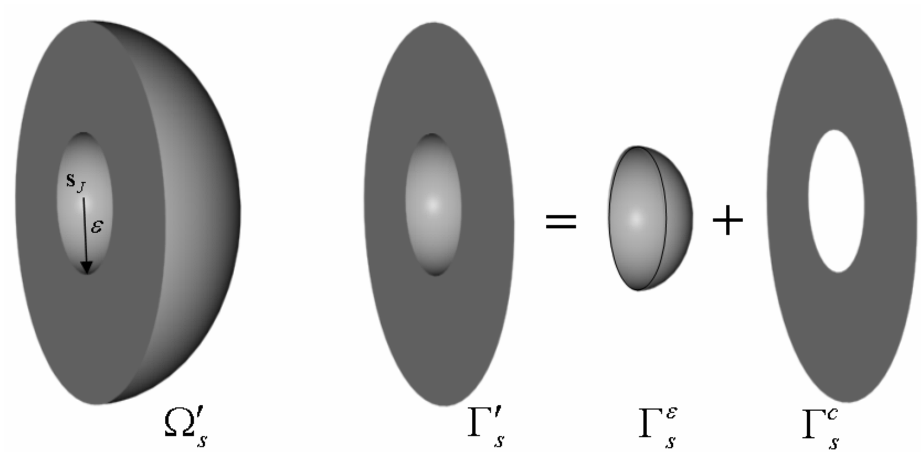
	Computational efficiency	Secondary results recovery	Accuracy	Stability	Convergence rate	Suitable for thin structures
Singular HBNM	Low	Complex	High	Good	High	No
Regular HBNM	High	Simple	High	Poor	Low	Yes

- Combining RHBNM and SHBNM in multi-domain analysis of complicated structures which contains very thin sub-structures (nanotube based composites, for example) is a promising topic of future research work



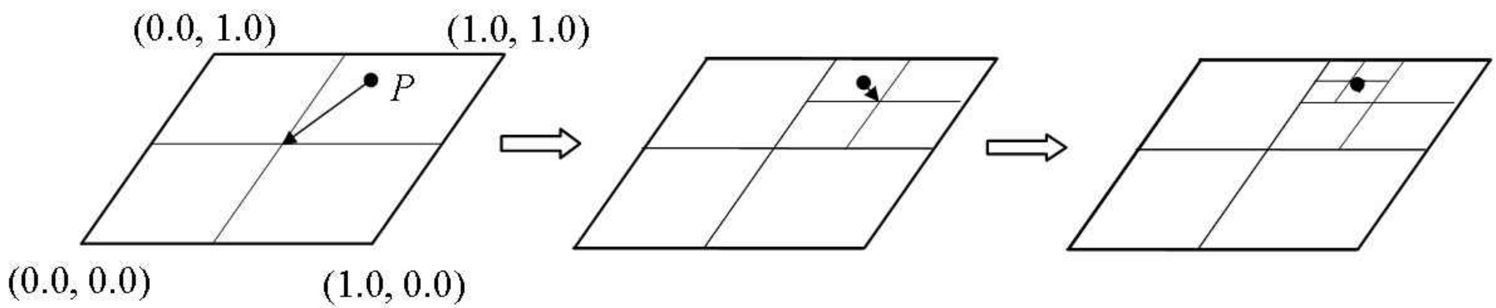
Inclusion problems


-1
-2
-3
-4
-5
$Log_{10}(e)$
0
2
4
6
8
10
12
14
Scale factor, *SF*
DM-u
DM-q
SF-u
SF-q



$$\lim_{\varepsilon \to 0} \int_{\Omega'_s} u_{,ii} v d\Omega = 0$$

$$\lim_{\varepsilon \to 0} \int_{\Gamma_s^\varepsilon} q v d\Gamma + \lim_{\varepsilon \to 0} \int_{\Gamma_s^c} q v d\Gamma - \lim_{\varepsilon \to 0} \int_{\Gamma_s^c} \tilde{q} v d\Gamma = 0$$


(0.0, 1.0)
(1.0, 1.0)
P
(0.0, 0.0)
(1.0, 0.0)